Expressive Power.

$\mathbf{BSML}$ and the Propositional Independence Logic $\mathbf{PL}(\perp, \mathbb{V})$

Søren Brinck Knudstorp

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Plan for the talk

- Background and Motivation
- Preliminaries
- Expressive Completeness of $\mathbf{BSML}$
- Expressive Completeness of $\mathbf{PL}(\bot, \mathbb{V})$
- Conclusion

Background and Motivation

- Aloni (2022) introduced $\mathbf{BSML}$, a modal team logic for modeling linguistic phenomena.
- Aloni, Anttila and Yang (2023) considered two extensions:

  - $||\mathbf{BSML}(\mathbin{\mathbb{V}})|| = \{$All properties$\}$
  - $||\mathbf{BSML}(\oslash)|| = \{$Union-closed properties$\}$
  - $||\mathbf{BSML}|| =?$

- The expressive power of $\mathbf{BSML}$'s propositional analogue $\mathbf{PL}(\text{NE})$ an open problem (Yang and Väänänen 2017).
- As is the expressive power of $\mathbf{PL}(\bot, \mathbin{\mathbb{V}})$.
- Today, we show that:
  1. $\mathbf{BSML}$ and $\mathbf{PL}(\text{NE})$ are expressively complete for all convex, union-closed properties.
  2. $\mathbf{PL}(\bot, \mathbin{\mathbb{V}})$ is expressively complete for all properties that are (i) <4-closed, (ii) $\downarrow$-singleton closed, and (iii) contains the empty team.

For the sake of simplicity, we only consider the propositional versions; i.e., $\mathrm{PL}(\text{NE})$ instead of BSML.

$\mathbf{PL}(\text{NE})$: Definition

Definition ($\mathbf{PL}(\text{NE})$: Syntax and team-semantic clauses)

Syntax of $\mathbf{PL}(\text{NE})$:

$$\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \text{NE}$$

Semantics: Let $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$. For $t \in \mathcal{P}(X)$, we define

$$\begin{array}{lll} t \vDash p & \text{iff} & \forall v \in t : v(p) = 1, \\ t \vDash \neg p & \text{iff} & \forall v \in t : v(p) = 0, \\ t \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \ \textit{and} \ s \vDash \psi, \\ t \vDash \varphi \vee \psi & \text{iff} & \text{there exist } t', t'' \in \mathcal{P}(X) \text{ such that } t' \vDash \varphi; \\ & & t'' \vDash \psi; \text{ and } t = t' \cup t'', \\ t \vDash \text{NE} & \text{iff} & t \neq \emptyset. \end{array}$$

OBS: From here on out, we fix a *finite* set of propositional letters Prop, considering teams over $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$.

$\mathbf{PL}(\text{NE})$: Expressive Power

Definition (Expressive completeness)

- A (team) property is a set of teams $S \subseteq \mathcal{P}(X)$.
- Observe that the 'truthset' of a formula φ, defines a property:

$$||\varphi|| := \{t \in \mathcal{P}(X) \mid t \vDash \varphi\} \subseteq \mathcal{P}(X).$$

- And that *a set of formulas* Γ defines *a set of properties*

$$||\Gamma|| := \{||\varphi|| \mid \varphi \in \Gamma\} \subseteq \mathcal{PP}(X).$$

- A logic (or language) $\mathcal{L}$ is expressively complete for a set of properties $\mathcal{S} \subseteq \mathcal{PP}(X)$:iff

$$||\mathcal{L}|| = \mathcal{S}$$

Theorem (expressive completeness of $\mathbf{PL}(\text{NE})$)

$$||\mathbf{PL}(\text{NE})|| = \{\text{property } S \subseteq \mathcal{P}(X) \mid S \text{ is convex and union closed}\}$$

$\mathbf{PL}(\textsc{ne})$: Proof of Expressive Power

Useful fact: $\mathbf{PL}$ is expressively complete for all downward- and union-closed properties containing the empty team.

Theorem (expressive completeness of $\mathbf{PL}(\textsc{ne})$)

$$||\mathbf{PL}(\textsc{ne})|| = \{\text{property } S \subseteq \mathcal{P}(X) \mid S \text{ is convex and union closed }\}$$

Proof.

'$\subseteq$':

- Union closure: By induction, already known ✓
- Convexity: By induction, *see blackboard*

'$\supseteq$': Let S be a convex, union-closed set of teams.

- If $\varnothing \in S$, then by convexity, it is downward closed, so done ✓
- If $\varnothing \notin S = \{t_1, \ldots, t_n\}$, we claim that $||\phi_S|| = S$ where

$$\phi_S := \bigvee \{\textsc{ne} \wedge (\chi_{v_1} \vee \cdots \vee \chi_{v_n}) \mid (v_1, ..., v_n) \in (t_1 \times \cdots \times t_n)\}$$

See blackboard □

Summary so far

Modal case:

- $||\mathbf{BSML}(\mathbin{\vee\mkern-7mu\vee})|| = \{$All properties$\}$
- $||\mathbf{BSML}(\oslash)|| = \{$Union-closed properties$\}$
- $||\mathbf{BSML}|| = \{$Convex and union-closed properties$\}$

Propositional case:

- $||\mathbf{PL}(\text{NE}, \mathbin{\vee\mkern-7mu\vee})|| = \{$All properties$\}$
- $||\mathbf{PL}(\text{NE}, \oslash)|| = \{$Union-closed properties$\}$
- $||\mathbf{PL}(\text{NE})|| = \{$Convex and union-closed properties$\}$

Question: *Can we find a logic/language expressively complete for all convex properties simpliciter?*

Answer: *Yes! Aleksi Anttila has found such!*

What about $||\mathbf{PL}(\bot, \mathbb{V})||$?

$\mathbf{PL}(\perp, \mathbb{V})$: Definition and Expressive Power

Definition ($\mathbf{PL}(\perp, \mathbb{V})$: Syntax and team-semantic clauses)

Syntax of $\mathbf{PL}(\perp, \mathbb{V})$:

$$\varphi ::= \ p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \mathbb{V} \varphi \mid \alpha \perp \beta,$$

where α, β are $\perp$-free.
Additional semantic clauses:

$t \vDash \varphi \mathbb{V} \psi$ iff $t \vDash \varphi$ or $t \vDash \psi$.
$t \vDash \alpha \perp \beta$ iff $\forall v, v' \in t \exists v'' \in t : [v(\alpha) = v''(\alpha)]$ and $[v'(\beta) = v''(\beta)]$.

Theorem (expressive completeness of $\mathbf{PL}(\perp, \mathbb{V})$)

$||\mathbf{PL}(\perp, \mathbb{V})|| = \{$property $S \subseteq \mathcal{P}(X)$ $|S$ contains $\varnothing$, is <4-closed and $\downarrow$-singleton closed$\}$

$\mathbf{PL}(\bot, \mathbb{V})$: Proof of Expressive Power

Theorem (expressive completeness of $\mathbf{PL}(\bot, \mathbb{V})$)

$$||\mathbf{PL}(\bot, \mathbb{V})|| = \{\text{property } S \subseteq \mathcal{P}(X) \mid S \text{ contains } \varnothing, \text{ is } {<}4\text{-closed and } \downarrow\text{-singleton closed}\}$$

Proof.

'$\subseteq$' follows by induction.
For '$\supseteq$', given $S \subseteq \mathcal{P}(X)$, let

$$\phi'_S := \mathbb{V}_{t \in S}\, \phi'_t,$$

where

$$\phi'_t = \begin{cases} \chi_t & \text{if } |t| < 4 \\ \chi_t \wedge \bigwedge_{t' \subsetneq t, |t'| \geq 2} [(\chi_v \vee \chi_{v_1}) \bot (\chi_v \vee \chi_{v_2})] & \text{if } |t| \geq 4 \end{cases},$$

where $v \in t \setminus t'$, $v_1 \neq v_2$ and $\{v_1, v_2\} \subseteq t'$.
One can then show that $||\phi'_S|| = S$. □

Conclusion

What we have done:

- Characterized the expressive power of $\mathbf{PL}(\text{NE})$ (and $\mathbf{BSML}$) and $\mathbf{PL}(\bot, \mathbb{V})$
- Obtained normal forms the logics of concern

What (might) come next:

- Writing paper with Aleksi Anttila on the convex logics
- Perhaps, trying to find a FO team logic expressively complete for all convex properties
  - Let us (me and Aleksi) know if you are interested in working on this! :)

Thank you!